Quantization on Generalized Heisenberg-Virasoro Algebra[1]

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Abstract: In a recent paper by the authors, Lie bialgebra structures on generalized Heisenberg-Virasoro algebra $\mathfrak{L}$ are considered. In this paper, the explicit formula of the quantization on generalized Heisenberg-Virasoro algebra is presented.

Key words: Lie bialgebras, quantization, generalized Heisenberg-Virasoro algebra, Hopf algebra.

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1 Introduction

In Hopf algebras or quantum groups theory, there are two standard methods to yield new bialgebras from old ones, one is twisting the product by a 2-cocycle but keeping the coproduct unchanged, another is twisting the coproduct by a Drinfel'd twist element but keeping the product unchanged. Constructing quantization of Lie bialgebras is an important method to produce new quantum groups (cf.[5],[6],[11], etc). Drinfel'd in [?] formulated a number of problems in quantum group theory, including the existence of a quantization for Lie bialgebras. In the paper [9] Etingof and Kazhdan gave a positive answer to some of Drinfel'd questions. In particular, they showed the existence of quantizations for Lie bialgebras, namely, any classical Yang-Baxter algebra can be quantized. Since then the interests in quantizations of Lie bialgebras have been growing in the mathematical literatures (e.g.,[8, 10, 13, 15]).

This Lie algebra is the universal central extension of the Lie algebra of differential operators on a circle of order at most one, which contains an infinite-dimensional Heisenberg subalgebra and the Virasoro subalgebra. The natural action of the Virasoro subalgebra on the Heisenberg subalgebra is twisted with a 2-cocycle. The structure and representation theory for the generalized Heisenberg-Virasoro algebra has been well developed (e.g., [1, 2, 12, 16, 20]). The structure of the irreducible highest weight modules and verma modules for the twisted Heisenberg-Virasoro algebra are determined in [1, 2].

Recently, the Lie bialgebra structures on generalized Heisenberg-Virasoro algebra $\mathfrak{L}$ was discussed in [3], which turned out the centerless generalized Heisenberg-Virasoro algebra $\overline{\mathfrak{L}}$ is triangular coboundary. We note that generalized Heisenberg-Virasoro algebra is Γ graded, where Γ is an abelian group over a field $\mathbb{F}$ of characteristic zero. For Γ and T is a vector space over field $\mathbb{F}$, the *generalized Heisenberg-Virasoro algebra* $\mathfrak{L} := \mathfrak{L}(\Gamma)$([17]) is a Lie algebra generated by

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$\{L_x = t^x\partial, I_x = t^x, C_L, C_I, C_{LI}, x \in \Gamma\}$, subject to the following relations:

$$\begin{aligned}
&[L_x, L_y] = (y-x)L_{x+y} + \delta_{x+y,0}\tfrac{1}{12}(x^3 - x)C_L,\\
&[I_x, I_y] = y\delta_{x+y,0}C_I,\\
&[L_x, I_y] = yI_{x+y} + \delta_{x+y,0}(x^2 - x)C_L,\\
&[\mathfrak{L}, C_L] = [\mathfrak{L}, C_I] = [\mathfrak{L}, C_{LI}] = 0.
\end{aligned} \tag{1.1}$$

The Lie algebra $\mathfrak{L}$ has a generalized Heisenberg subalgebra and a generalized Virasoro subalgebra interwined with a 2-cocycle. Set $\mathfrak{L}_x = \mathrm{Span}_{\mathbb{F}}\{L_x, I_x\}$ for $x \in \Gamma\backslash\{0\}$, $\mathfrak{L}_0 = \mathrm{Span}_{\mathbb{F}}\{L_0, I_0, C_L, C_I, C_{LI}\}$. Then $\mathfrak{L} = \underset{x\in\Gamma}{\oplus} \mathfrak{L}_x$ is a graded Lie algebra. Denote C the center of $\mathfrak{L}$, then $\mathcal{C} = \mathrm{Span}_{\mathbb{F}}\{I_0, C_L, C_I, C_{LI}\}$. Denote $\overline{\mathfrak{L}} = \mathfrak{L}/\mathcal{C}$, then $\overline{\mathfrak{L}}$ is the centerless generalized Heisenberg-Virasoro algebra.

The main result of this paper is the following theorem:

Theorem 1.1. *We choose two distinguished elements $h = \alpha^{-1}L_0$ and $e = I_\alpha$ with $\alpha \in \Gamma \setminus \{0\}$, such that $[h, e] = e$ in $\overline{\mathfrak{L}}$, there exists a structure of noncommutative and noncocommutative Hopf algebra $(U(\overline{\mathfrak{L}})[[t]], m, \iota, \Delta, S, \epsilon)$ on $U(\overline{\mathfrak{L}})[[t]]$ with $U(\overline{\mathfrak{L}})[[t]]/tU(\overline{\mathfrak{L}})[[t]] \cong U(\overline{\mathfrak{L}})$, which preserves the product and counit of $U(\overline{\mathfrak{L}})[[t]]$ but with a comultiplication and antipode defined by:*

$$\begin{aligned}
&\Delta(L_\beta) = 1 \otimes L_\beta + L_\beta \otimes (1-et)^{\alpha^{-1}\beta} + \alpha h^{(1)} \otimes (1-et)^{-1} I_{\alpha+\beta}t,\\
&\Delta(I_\alpha) = 1 \otimes I_\alpha + I_\alpha \otimes (1-et),\\
&S(L_\beta) = -(1-et)^{-\alpha^{-1}\beta}L_\beta + (1-et)^{-\alpha^{-1}\beta} h^{[1]}_{-\alpha^{-1}\beta} I_{\alpha+\beta}t,\\
&S(I_\alpha) = -(1-et)^{-1}I_\alpha.
\end{aligned}$$

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2 Preliminaries

In this section, we summarize some basic definitions and results concerning Lie bialgebra structures which will be used in the following discussions. For a detailed discussion of this subject we refer the reader to the literatures (see [3] and references therein).

Let $\overline{\mathfrak{L}}$ be the centerless generalized Heisenberg-Virasoro algebra and $U(\overline{\mathfrak{L}})$ the universal enveloping algebra of $\overline{\mathfrak{L}}$. Then $U(\overline{\mathfrak{L}})$ is equiped with a natural Hopf algebraic structure $(U(\overline{\mathfrak{L}}), m, \iota, \Delta_0, S_0, \epsilon)$, i.e.,

$$\Delta_0(X) = X \otimes 1 + 1 \otimes X, \quad S_0(X) = -X, \quad \epsilon(X) = 0, \tag{2.1}$$

where Δ_0 is a comultiplication, ϵ is a counit and S_0 is an antipode. In particular,

$$\Delta_0(1) = 1 \otimes 1 \quad \text{and} \quad \epsilon(1) = S_0(1) = 1. \tag{2.2}$$

The following result is due to W. Michaelis(see [18]).

Theorem 2.1. *Let L be a Lie algebra containing two linear independent elements a and b satisfying $[a, b] = kb$ with $0 \neq k \in \mathbb{F}$. Set*

$$r = a \otimes b - b \otimes a$$

and define a linear map

$$\Delta_r : L \to L \otimes L$$

by setting

$$\Delta_r(x) = x \cdot r = [x, a] \otimes b - b \otimes [x, a] + a \otimes [x, b] - [x, b] \otimes a, \quad \forall\ x \in L.$$

Then Δ_r equips L with structure of a triangular coboundary Lie bialgebra.

An algebra L equipped with a classical Yang-Baxter r-matrix r is called a *classical Yang-Baxter algebra*. It was shown in [9] that any classical Yang-Baxter algebra can be quantized.

Definition 2.2. Let $(H, m, \iota, \Delta_0, S_0, \epsilon)$ be a Hopf algebra over a commutative ring R. A Drinfel'd twist $\mathcal{F}$ on H is an invertible element of $H \otimes H$ such that

$$(\mathcal{F} \otimes 1)(\Delta_0 \otimes \mathrm{Id})(\mathcal{F}) = (1 \otimes \mathcal{F})(\mathrm{Id} \otimes \Delta_0)(\mathcal{F}),$$
$$(\epsilon \otimes \mathrm{Id})(\mathcal{F}) = 1 \otimes 1 = (\mathrm{Id} \otimes \epsilon)(\mathcal{F}).$$

The following result is well known (see e.g., [D1], [ES]).

Lemma 2.3. *Let $(H, m, \iota, \Delta_0, S_0, \epsilon)$ be a Hopf algebra over a commutative ring and $\mathcal{F}$ be a Drinfel'd twist on H, then $w = m(\mathrm{Id} \otimes S_0)(\mathcal{F})$ is invertible in H with $w^{-1} = m(S_0 \otimes \mathrm{Id})(\mathcal{F}^{-1})$. Moreover, define Δ :$H \to H \otimes H$ and S :$H \to H$ by*

$$\Delta(x) = \mathcal{F}\Delta_0(x)\mathcal{F}^{-1}, \quad S = wS_0(x)w^{-1}, \quad \forall\ x \in H.$$

Then $(H, m, \iota, \Delta, S, \epsilon)$ is a new Hopf algebra, which is said to be the twisting of H by the Drinfel'd twist $\mathcal{F}$.

Let $\mathbb{F}[[t]]$ be a ring of formal power series. Assume that L is a triangular Lie bialgebra with a classical Yang-Baxter r-matrix r. Denote by $U(L)$ the universal enveloping algebra of L, with the standard Hopf algebra structure $(U(L), m, \iota, \Delta_0, S_0, \epsilon)$. Now consider the topologically free $F[[t]]$-algebra $U(L)[[t]]$ (see [p.4] [11]), which can be viewed as an associative $\mathbb{F}$-algebra of formal power series with coefficients in $U(L)$. Naturally, $U(L)[[t]]$ is equiped with an induced Hopf algebra structure arising from that on $U(L)$. By abuse of notation, we denote it by $(U(L)[[t]], m, \iota, \Delta_0, S_0, \epsilon)$.

An algebra A equipped with a classical Yang-Baxter r-matrix r is called a *classical Yang-Baxter algebra*. It is showed in [9] that any classical Yang- Baxter algebra can be quantized.

For any element x of a unital R-algebra (R is a ring) and $a \in R$, we set (see, e.g., [14])

$$x_a^{(n)} := (x+a)(x+a+1)\cdots(x+a+n-1),$$
$$x_a^{[n]} := (x+a)(x+a-1)\cdots(x+a-n+1),$$

and $x^{(n)} := x_0^{(n)}$, $x^{[n]} := x_0^{[n]}$.

Lemma 2.4. (see [13, 14]) *For any element x of a unital $\mathbb{F}$-algebra, $a, b \in \mathbb{F}$, and $r, s, t \in \mathbb{Z}$, one has*

$$x_a^{(s+t)} = x_a^{(s)} x_{a+s}^{(t)}, \qquad x_a^{[s+t]} = x_a^{[s]} x_{a-s}^{[t]}, \qquad x_a^{[s]} = x_{a-s+1}^{(s)}, \tag{2.3}$$

$$\sum_{s+t=r} \frac{(-1)^t}{s!t!} x_a^{[s]} x_b^{(t)} = \binom{a-b}{r} = \frac{(a-b)\cdots(a-b-r+1)}{r!}, \tag{2.4}$$

$$\sum_{s+t=r} \frac{(-1)^t}{s!t!} x_a^{[s]} x_{b-s}^{[t]} = \binom{a-b+r-1}{r} = \frac{(a-b)\cdots(a-b+r-1)}{r!}. \tag{2.5}$$

The following popular result will be frequently used in the third part of this paper.

Lemma 2.5. (see e.g., [Proposition 1.3(4)][19]) *For any elements x, y of an associative algebra A, and $m \in \mathbb{Z}_+$, one has*

$$xy^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} y^{m-k} (\mathrm{ad}\, y)^k (x). \tag{2.6}$$

3 Proof of main results

In this section, assume that $\mathfrak{L}$ is the generalized Heisenberg-Virasoro algebra defined in (1.1). We have obtained that the Lie bialgebra structures on centerless generalized Heisenberg-Virasoro algebra $\overline{\mathfrak{L}}$ are triangular coboundary, namely, there always exist solutions of CYBE in $\overline{\mathfrak{L}} \otimes \overline{\mathfrak{L}}$. Therefore, $\overline{\mathfrak{L}}$ can be quantized by the above arguments. In what follows we will use a Drinfel'd twist (see Definition 2.2) to proceed the quantization on centerless generalized Heisenberg-Virasoro algebra $\overline{\mathfrak{L}}$.

To describe quantizations of $U(\overline{\mathfrak{L}})$, we need to construct explicitly Drinfeld twists according to Lemma 2.3. Set

$$h := \alpha^{-1} L_0, \qquad e := I_\alpha$$

for a fixed $\alpha \in \Gamma \backslash \{0\}$. It is easily to see $[h, e] = e$ by (1.1). Then it follows from Theorem 2.1 that $r = h \otimes e - e \otimes h$ is a solution of CYBE, namely, r is a classical r-matrix. Now we can use this r-matrix determined by e and h to construct a Drinfel'd twist. This will be done by several lemmas.

Lemma 3.1. *For $a \in \mathbb{F}$, $i \in \mathbb{Z}_+$, $n \in \mathbb{Z}$, $\beta \in \Gamma$ and $\alpha \in \Gamma \backslash \{0\}$, one has*

$$\begin{array}{ll} L_\beta h_a^{(i)} = h_{a-\alpha^{-1}\beta}^{(i)} L_\beta, & L_\beta h_a^{[i]} = h_{a-\alpha^{-1}\beta}^{[i]} L_\beta, \\ I_\alpha h_a^{(i)} = h_{a-1}^{(i)} I_\alpha, & I_\alpha h_a^{[i]} = h_{a-1}^{[i]} I_\alpha, \\ e^n h_a^{(i)} = h_{a-n}^{(i)} e^n, & e^n h_a^{[i]} = h_{a-n}^{[i]} e^n. \end{array}$$

Proof. We only prove the first equation (the others can be obtained similarly). Since $L_\beta h - hL_\beta = -\alpha^{-1}\beta L_\beta$, there is nothing to prove for $i = 1$. For the induction step, suppose that it holds for i, then one has

$$\begin{aligned} L_\beta h_a^{(i+1)} &= L_\beta h_a^{(i)}(h + a + i) \\ &= h_{a-\alpha^{-1}\beta}^{(i)} L_\beta (h + a + i) \\ &= h_{a-\alpha^{-1}\beta}^{(i)} (h - \beta + a + i) L_\beta \\ &= h_{a-\alpha^{-1}\beta}^{(i+1)} L_\beta. \end{aligned}$$

Now for $a \in \mathbb{F}$, set

$$\mathcal{F}_a = \sum_{r=0}^{\infty} \tfrac{(-1)^r}{r!} h_a^{[r]} \otimes e^r t^r, \quad F_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{(r)} \otimes e^r t^r, \tag{3.1}$$

$$u_a = m \cdot (S_0 \otimes \mathrm{Id})(F_a), \quad v_a = m \cdot (\mathrm{Id} \otimes S_0)(\mathcal{F}_a).$$

Write $\mathcal{F} = \mathcal{F}_0$, $F = F_0$, $u = u_0$, $v = v_0$. Since $S_0(h_a^{(r)}) = (-1)^r h_{-a}^{[r]}$ and $S_0(e^r) = (-1)^r e^r$, we have

$$u_a = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} h_{-a}^{[r]} e^r t^r, \qquad v_a = \sum_{r=0}^{\infty} \frac{1}{r!} h_a^{[r]} e^r t^r. \tag{3.2}$$

Lemma 3.2. *For $a, b \in \mathbb{F}$, one has*

$$\mathcal{F}_a F_b = 1 \otimes (1 - et)^{a-b}, \qquad v_a u_b = (1 - et)^{-(a+b)}.$$

Proof. Using the equations (2.4) and (3.1), we have

$$\begin{aligned} \mathcal{F}_a F_b &= \sum_{r,s=0}^{\infty} \frac{(-1)^r}{r!s!} h_a^{[r]} h_b^{(s)} \otimes e^r e^s t^r t^s \\ &= \sum_{m=0}^{\infty} (-1)^m \Big(\sum_{r+s=m} \frac{(-1)^s}{r!s!} h_a^{[r]} h_b^{(s)} \Big) \otimes e^m t^m \\ &= \sum_{m=0}^{\infty} (-1)^m \binom{a-b}{m} \otimes e^m t^m \\ &= 1 \otimes (1 - et)^{a-b}. \end{aligned}$$

From (2.5), (3.2) and Lemma 3.1, we obtain that

$$
\begin{aligned}
v_a u_b &= \sum_{r,s=0}^{\infty} \frac{(-1)^s}{r!s!} h_a^{[r]} e^r h_{-b}^{[s]} e^s t^{r+s} \\
&= \sum_{m=0}^{\infty} \sum_{r+s=m} \frac{(-1)^s}{r!s!} h_a^{[r]} h_{-b-r}^{[s]} e^m t^m \\
&= \sum_{m=0}^{\infty} \binom{a+b+m-1}{m} e^m t^m \\
&= (1-et)^{-(a+b)}.
\end{aligned}
$$

Corollary 3.3. *For any $a \in \mathbb{F}$, the elements F_a and u_a are invertible with $F_a^{-1} = \mathcal{F}_a$, $u_a^{-1} = v_{-a}$. In particular, $F^{-1} = \mathcal{F}$, $u^{-1} = v$.*

Lemma 3.4. *For any $a \in \mathbb{F}$ and $r \in \mathbb{Z}_+$, one has $\Delta_0(h^{[r]}) = \sum_{i=0}^{r} \binom{r}{i} h_{-a}^{[i]} \otimes h_a^{[r-i]}$. In particular, one has $\Delta_0(h^{[r]}) = \sum_{i=0}^{r} \binom{r}{i} h^{[i]} \otimes h^{[r-i]}$.*

Proof. Since $\Delta_0(h) = 1 \otimes h + h \otimes 1$, it is easy to see that the result is true for $r = 1$. Suppose it is true for r, then for $r+1$, we have

$$
\begin{aligned}
\Delta_0(h^{[r+1]}) &= \Delta_0(h^{[r]}(h-r)) \\
&= \Delta_0(h^{[r]})\big(\Delta_0(h) - \Delta_0(r)\big) \\
&= \Big(\sum_{i=0}^{r} \binom{r}{i} h_{-a}^{[i]} \otimes h_a^{[r-i]}\Big)\big((h-r)\otimes 1 + 1 \otimes (h-r) + r(1\otimes 1)\big) \\
&= \Big(\sum_{i=1}^{r-1} \binom{r}{i} h_{-a}^{[i]} \otimes h_a^{[r-i]}\Big)\big((h-r)\otimes 1 + 1 \otimes (h-r)\big) + r\sum_{i=0}^{r} \binom{r}{i} h_{-a}^{[i]} \otimes h_a^{[r-i]} \\
&\quad + h_{-a}^{[r+1]} \otimes 1 + h_{-a}^{[r]} \otimes a + h_{-a}^{[r]} \otimes (h-r) + (h-r) \otimes h_a^{[r]} + 1 \otimes h_a^{[r+1]} - a \otimes h_a^{[r]} \\
&= 1 \otimes h_a^{[r+1]} + h_{-a}^{[r+1]} \otimes 1 + r\sum_{i=1}^{r-1} \binom{r}{i} h_{-a}^{[i]} \otimes h_a^{[r-i]} + h_{-a}^{[r]} \otimes (h+a) \\
&\quad + (h-a) \otimes h_a^{[r]} + \sum_{i=1}^{r-1} \binom{r}{i} h_{-a}^{[i+1]} \otimes h_a^{[r-i]} + \sum_{i=1}^{r-1} (-r+a+i) \binom{r}{i} h_{-a}^{[i]} \otimes h_a^{[r-i]} \\
&\quad + \sum_{i=1}^{r-1} \binom{r}{i} h_{-a}^{[i]} \otimes h_a^{[r-i+1]} + \sum_{i=1}^{r-1} (-a-i) \binom{r}{i} h_{-a}^{[i]} \otimes h_a^{[r-i]} \\
&= 1 \otimes h_a^{[r+1]} + h_{-a}^{[r+1]} \otimes 1 + \sum_{i=1}^{r} \left[\binom{r}{i-1} + \binom{r}{i}\right] h_{-a}^{[i]} \otimes h_a^{[r-i+1]} \\
&= \sum_{i=0}^{r+1} \binom{r+1}{i} h_{-a}^{[i]} \otimes h_a^{[r+1-i]}.
\end{aligned}
$$

Hence, the formula holds by induction.

Lemma 3.5. *The element* $\mathcal{F}=\sum\limits_{r=0}^{\infty}\frac{(-1)^r}{r!}h^{[r]}\otimes e^r t^r$ *is a Drinfeld twist on* $U(\overline{\mathfrak{L}})[[t]]$.

Proof. It can be proved directly by the similar methods as those presented in the proof of [Proposition 2.5][15].

Now we can now perform the process of twisting the standard Hopf structure $(U(\overline{\mathfrak{L}}),m,\iota,\Delta_0,S_0,\epsilon)$ defined in (2.1) by the Drinfel'd twist $\mathcal{F}$ constructed above. The following lemmas are very useful to our main results.

Lemma 3.6. *For* $a\in\mathbb{F}$, $\beta\in\Gamma$, $\alpha\in\Gamma\backslash\{0\}$, *one has*

$$\begin{aligned}(L_\beta\otimes 1)F_a&=F_{a-\alpha^{-1}\beta}(L_\beta\otimes 1),\\ (I_\alpha\otimes 1)F_a&=F_{a-1}(I_\alpha\otimes 1).\end{aligned}$$

Proof. It follows directly from equation (3.1) and Lemma 3.1.

Lemma 3.7. *For* $a\in\mathbb{F}$, $\beta\in\Gamma$, $\alpha\in\Gamma\backslash\{0\}$ *and* $r\in\mathbb{Z}_+$, *one has*

$$L_\beta e^r=e^rL_\beta+\alpha re^{r-1}I_{\alpha+\beta}, \tag{3.3}$$

$$I_\alpha e^r=e^rI_\alpha. \tag{3.4}$$

Proof. By Lemma 2.5 and equation (1.1), we have

$$\begin{aligned}L_\beta e^r&=\sum_{i=0}^{r}(-1)^i\binom{r}{i}e^{r-i}(\mathrm{ad}\,e)^i(L_\beta)\\ &=e^rL_\beta+\alpha re^{r-1}I_{\alpha+\beta}.\end{aligned}$$

Similarly, one can get (3.4).

Lemma 3.8. *For* $a\in\mathbb{F}$, $\beta\in\Gamma,\alpha\in\Gamma\backslash\{0\}$, *we have*

$$(1\otimes L_\beta)F_a=F_a(1\otimes L_\beta)+\alpha F_{a+1}(h_a^{(1)}\otimes I_{\alpha+\beta}t), \tag{3.5}$$

$$(1\otimes I_\alpha)F_a=F_a(1\otimes I_\alpha). \tag{3.6}$$

Proof. By equations (2.3), (3.1) and (3.3), one has

$$\begin{aligned}(1\otimes L_\beta)F_a&=\sum_{r=0}^{\infty}\tfrac{1}{r!}h_a^{(r)}\otimes L_\beta e^rt^r\\ &=\sum_{r=0}^{\infty}\tfrac{1}{r!}h_a^{(r)}\otimes(e^rL_\beta+\alpha re^{r-1}I_{\alpha+\beta})t^r\end{aligned}$$

$$= F_a(1\otimes L_\beta) + \sum_{r=1}^{\infty} \tfrac{\alpha}{(r-1)!} h_a^{(r)} \otimes e^{r-1} I_{\alpha+\beta} t^r$$

$$= F_a(1\otimes L_\beta) + \sum_{r=0}^{\infty} \tfrac{\alpha}{r!} h_a^{(r+1)} \otimes e^{r} I_{\alpha+\beta} t^{r+1}$$

$$= F_a(1\otimes L_n) + \sum_{r=0}^{\infty} \tfrac{\alpha}{r!} h_{a+1}^{(r)} h_a^{(1)} \otimes e^{r} I_{\alpha+\beta} t^{r+1}$$

$$= F_a(1\otimes L_n) + \alpha F_{a+1}\big(h_a^{(1)} \otimes I_{\alpha+\beta} t\big).$$

This proves equation (3.5). Similarly, (3.6) follow from (3.4) .

Lemma 3.9. *For $a\in\mathbb{F}$, $\beta\in\Gamma, \alpha\in\Gamma\backslash\{0\}$, one has*

$$L_\beta u_a = u_{a+\alpha^{-1}\beta} L_\beta - u_{a+\alpha^{-1}\beta} h^{[1]}_{-a-\alpha^{-1}\beta} I_{\alpha+\beta} t, \tag{3.7}$$

$$I_\alpha u_a = u_{a+1} I_\alpha. \tag{3.8}$$

Proof. From equations (2.3), (3.2), (3.3) and Lemma 3.1, one has

$$L_\beta u_a = \sum_{r=0}^{\infty} \tfrac{(-1)^r}{r!} L_\beta h^{[r]}_{-a} e^r t^r$$

$$= \sum_{r=0}^{\infty} \tfrac{(-1)^r}{r!} h^{[r]}_{-a-\alpha^{-1}\beta} L_\beta e^r t^r$$

$$= \sum_{r=0}^{\infty} \tfrac{(-1)^r}{r!} h^{[r]}_{-a-\alpha^{-1}\beta} \big(e^r L_\beta + \alpha r e^{r-1} I_{\alpha+\beta}\big) t^r$$

$$= u_{a+\alpha^{-1}\beta} L_\beta + \sum_{r=1}^{\infty} \tfrac{(-1)^r \alpha}{(r-1)!} h^{[r]}_{-a-\alpha^{-1}\beta} e^{r-1} I_{\alpha+\beta} t^r$$

$$= u_{a+\alpha^{-1}\beta} L_\beta - \sum_{r=0}^{\infty} \tfrac{(-1)^r}{r!} h^{[r+1]}_{-a-\alpha^{-1}\beta} e^{r} I_{\alpha+\beta} t^{r+1}$$

$$= u_{a+\alpha^{-1}\beta} L_\beta - \sum_{r=0}^{\infty} \tfrac{(-1)^r}{r!} h^{[r]}_{-a-\alpha^{-1}\beta} h^{[1]}_{-a-\alpha^{-1}\beta-r} e^{r} I_{\alpha+\beta} t^{r+1}$$

$$= u_{a+\alpha^{-1}\beta} L_\beta - \sum_{r=0}^{\infty} \tfrac{(-1)^r}{r!} h^{[r]}_{-a-\alpha^{-1}\beta} e^{r} h^{[1]}_{-a-\alpha^{-1}\beta} I_{\alpha+\beta} t^{r+1}$$

$$= u_{a+\alpha^{-1}\beta} L_\beta - u_{a+\alpha^{-1}\beta} h^{[1]}_{-a-\alpha^{-1}\beta} I_{\alpha+\beta} t.$$

Hence, (3.7) holds. Similarly, one can get (3.8) by Lemma 3.1 and Lemma 3.7.

Now we have enough in hand to prove our main theorem in this paper.

Proof of Theorem 1.1. By Lemma 2.3, Lemma3.2, Corollary 3.3, Lemma3.6 and Lemma3.8, we have

$$\Delta(L_\beta) = \mathcal{F}\Delta_0(L_n)\mathcal{F}^{-1}$$

$$= \mathcal{F}(L_\beta\otimes 1)F + \mathcal{F}(1\otimes L_\beta)F$$

$$
\begin{aligned}
&= \mathcal{F}F_{-\alpha^{-1}\beta}(L_\beta\otimes 1)+\mathcal{F}\big(F(1\otimes L_\beta)+\alpha F_1(h^{(1)}\otimes I_{\alpha+\beta})t\big)\\
&= L_\beta\otimes(1-et)^{\alpha^{-1}\beta}+1\otimes L_\beta+\alpha h^{(1)}\otimes(1-et)^{-1}I_{\alpha+\beta}t\\
&= 1\otimes L_\beta+L_\beta\otimes(1-et)^{\alpha^{-1}\beta}+\alpha h^{(1)}\otimes(1-et)^{-1}I_{\alpha+\beta}t.
\end{aligned}
$$

$$
\begin{aligned}
\Delta(I_\alpha) &= \mathcal{F}\Delta_0(I_\alpha)\mathcal{F}^{-1}\\
&= \mathcal{F}(I_\alpha\otimes 1)F+\mathcal{F}(1\otimes I_\alpha)F\\
&= \mathcal{F}F_{-1}(I_\alpha\otimes 1)+\mathcal{F}F(1\otimes I_\alpha)\\
&= 1\otimes I_\alpha+I_\alpha\otimes(1-et).
\end{aligned}
$$

Again by Lemma 2.3, Lemma3.2, Corollary 3.3 and Lemma 3.9, we have

$$
\begin{aligned}
S(L_\beta) &= u^{-1}S_0(L_\beta)u\\
&= -vL_\beta u\\
&= -v(u_{\alpha^{-1}\beta}L_\beta-u_{\alpha^{-1}\beta}h^{[1]}_{-\alpha^{-1}\beta}I_{\alpha+\beta}t)\\
&= -(1-et)^{-\alpha^{-1}\beta}L_\beta+(1-et)^{-\alpha^{-1}\beta}h^{[1]}_{-\alpha^{-1}\beta}I_{\alpha+\beta}t.\\
S(I_\alpha) &= u^{-1}S_0(I_\alpha)u\\
&= -vI_\alpha u\\
&= -vu_1I_\alpha\\
&= -(1-et)^{-1}I_\alpha.
\end{aligned}
$$

So the proof is complete!

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References

[1] E. Arbarello, C. De Concini, V.G. Kac, C. Procesi, Moduli spaces of curves and representation theory, Comm. Math. Phys., **117**, 1-36(1988)

[2] Y. Billig, Respresentations of the twisted Heisenberg-Virasoro algebra at level zero, Canad. Math. Bulletin, **46**, 529-537(2003)

[3] H. Chen, R. Shen, J, Zhang, Lie bialgebra structures on generalized Heisenberg-Virasoro algebra, in press.

[4] V.G. Drinfel'd, Hamlitonian structures on Lie group,Lie algebras and the geometric meaning of classical Yang-Baxter equations, *Soviet Math. Dokl.* **27**, 68-71(1983)

[5] V.G. Drinfel'd, Quantum groups, in: Proceeding of the International Congress of Mathematicians, Vol. 1, 2, Berkeley, Calif. 1986, Amer. Math. Soc., Providence, RI, pp. 798-820, 1987

[6] V.G. Drinfel'd. Constant quasiclassical solutions of the Yang-Baxter quantum equation. Soviet Mathematics Doklady, **28**, 667C671(1983)

[7] V.G. Drinfel'd, On some unsolved problems in quantum group theory, *Lecture Notes in Mathematics*, **1510** , 1–8(1992)

[8] B. Entiquez, G. Halbout, Quantization of Γ-Lie bialgebras, *J. Algebra*, **319**, 3752–3769(2008)

[9] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras I, *Selecta Math. (New Series)*, **2**, 1–41(1996)

[10] P. Etingof, D. Kazhdan, Quantization of Lie bialgebras, part VI: Quantization of generalized Kac-Moody algealgebras,*Transformation Groups*, **13**, 527–539(2008)

[11] P. Etingof, O. Schiffmann, Lectures on Quantum groups, 2nd ed., Boston: International Press, 2002

[12] M.A. Fabbri, F. Okoh, Representations of Virasoro-Heisenberg algebras and Virasoro-toroidal algebras, *Canad. J. Math.*, **51**, 523-545(1999)

[13] C. Grunspan, Quantizations of the Witt algebra and of simple Lie algebras in characteristic p, *J. Algebra*, **280**, 145–161(2004)

[14] A. Giaquinto, J. Zhang, Bialgebra actin, twists and universal deformation formulas, *J. Pure Appl. Algebra*, **128**, 133–151(1998)

[15] N. Hu, X. Wang, Quantizations of generalized-Witt algebra anf of Jacobson-Witt algebra in the modular case, *J. Algebra*, **312**, 902–929(2007)

[16] Q. Jiang, C. Jiang, Representations of the twisted Heisenberg-Virasoro algebra and the full toroidal Lie algebras, Algebra Colloq., **14**, 117-134(2007)

[17] D. Liu, L. Zhu, Generalized Heisenberg-Virasoro algebras, *Frontiers of Mathematics in China*, **4**, 297-310(2009)

[18] W. Michaelis, Lie coalgebras, *Adv. in Math.*, **38**, 1–54(1980)

[19] H. Strade, R. Farnsteiner, *Modular Lie Algebras and Their Representations*, Monogr. Textbooks Pure Appl. Math., vol.116, Marcel Dekker (1988).

[20] R. Shen, C. Jiang, Derivation algebra and automorphism group of the twisted Heisenberg-Virasoro algebra, *Comm. Alg*, **34**, 2547-2558(2006)

[21] R. Shen, Q. Jiang, Y. Su, Verma modules over the generalized Heisenberg-Virasoro algebra, *Comm. Alg*, **36**, 1464-1473(2008)

[22] G. Song, Y. Su, Y. Wu, Quantization of generalized Virasoro-like algebras, *Linear Algebra and its Applications*, **428**, 2888–2899(2008)

[23] Y. Su, L. Yuan, Quantization of Schrödinger-Virasoro Lie algebra, *Front. Math. China*, **5**, 701–715(2010)